Physical and mathematical models of the greenhouse effect

My Research Questions

If the Earth had no atmosphere, its average surface temperature would be about –18°C. However, the heat trapping effect of the atmosphere, called the **greenhouse effect**, means that a dynamic equilibrium occurs around 14°C, and thus we can live a sustainable life on earth.(1)

The visible and short-infrared radiation from the sun passes easily through the atmosphere and warms the earth's surface. However, being cooler than the sun, the earth's surface radiates back at a longer infrared wavelength. Molecules of water vapor and carbon dioxide in the atmosphere absorb some of this radiation. These then emit infrared radiation in all directions, including back towards the earth. The atmosphere and the earth's surface both warm up until a higher equilibrium temperature is reached. As more of the atmosphere absorbs and re-radiates heat, the overall temperature of the earth increases. This effect is called **global-warming**.

By burning fossil fuels, industrial societies like Western Europe and the Americas are putting carbon dioxide into the atmosphere at a faster rate than plants can absorb it. This is adding to the greenhouse effect, hence an **enhanced greenhouse effect** occurs, and this may be causing global warming. There is strong evidence that the average temperature of the earth is increasing. Although the effects of global warming cannot be predicted with any certainty, this phenomenon is worthy of further study. Moreover, when we covered the physics topic 8.2 "Thermal energy transfer" we learned the basics of the greenhouse effect, the enhanced greenhouse effect, the energy balance in the earth surface-atmosphere system, climate change and (my biggest concern) global warming. My passion for saving the environment and education people about the dangers of global warming have been developed in my physics IA project. I had two approaches in mind, and my teacher encouraged me to follow both physical and mathematical models of the greenhouse effect.

The purpose of my physics exploration is two-fold. First, I want to demonstrate global warming by a physical model. This will consist of two large soda bottles, one with flat soda and another with fizzy soda. The fizzy soda will produce an atmosphere with CO_2 while the flat soda will not. Both bottles are then set in direct sunlight for an hour and a record of the temperatures of each are recorded. The CO_2 atmosphere bottle ends up with a higher equilibrium temperature, thus demonstrating the greenhouse effect, the enhanced greenhouse effect and so demonstrating global warming.

The second part of this exploration is to produce a simple one-dimensional mathematical model of the atmosphere. This is done in Excel. The various parameters affecting the balanced or equilibrium temperature are explored.

A PHYSICAL MODEL OF THE GREENHOUSE EFFECT

My physical model of the greenhouse effect consisted of two identical plastic bottles set in the sun but one bottle had flat soda in it and the other had fresh, fizzy soda producing an atmosphere of CO_2 in it.[3] I then measured the temperature over time and observed the effect of the different atmospheres on the absorption of heat from the sun.

I took two identical 2-liter clear plastic soda drink bottles and fitted them each with thermometer probes connected to a Vernier's *LabPro* interface data logger unit and then into my computer using Vernier's *LoggerPro* graphing software.[4]

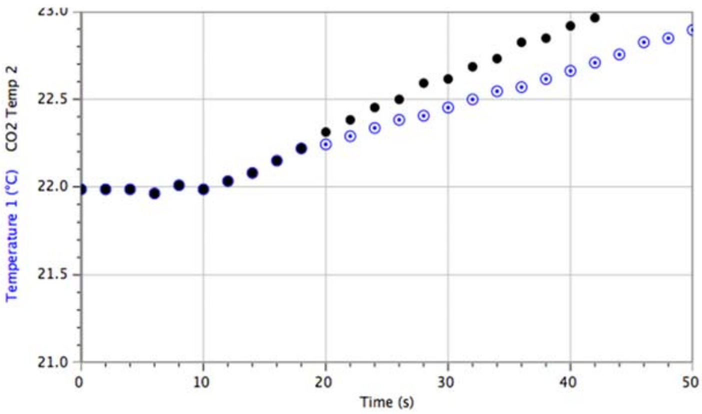
Next I took two identical 12 once bottles of soda at room temperature and opened one and poured it into a large mixing bowl. I agitated the soda until all the fizz was gone. This was the flat or non-CO_2 soda. Using a funnel I filled one of the 2-litre bottles with the flat soda and then opened a fresh bottle of soda and poured it into the second 2-litre bottle. Both bottles were situated in the direct sunlight. I started data logging and sealed the bottle caps with the thermometer leads passing through a small hole in the lid. I recorded the temperature every two seconds for about an hour.

As expected, I found that the CO_2 bottle retained slightly more of the radiant energy from the sun and hence had a higher temperature. Graph 1 is a close up of the first minute of data, and as you can see there was no temperature change for about the first ten seconds. It took this long to prepare the bottles. Soon after this the temperatures of both bottles started rising, and by 20 seconds the CO_2 bottle was getting warmer the non-$C0_2$ bottle.

Photograph of the setup.

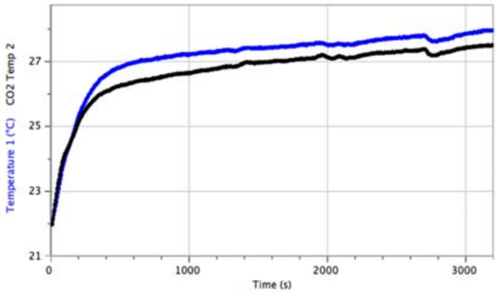
Above:
Photograph of experimental setup, outside in the sun.

Right:
Graph 1, A Close Up of Temperature and time for the first 50 seconds.

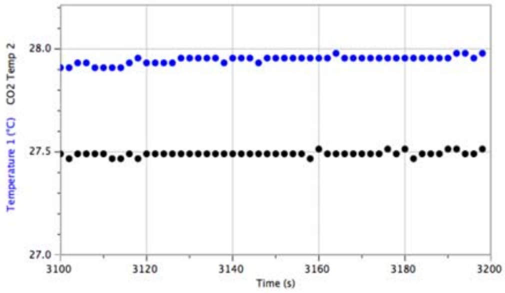


Graph 2 (below) is temperature against time for about an hour. The CO_2 bottle is always slightly warmer than the flat soda bottle. Eventually they both reach equilibrium, but not at the same temperature. The CO_2 bottle remains about one-half a degree higher compared to the non-$C0_2$ bottle. The small blip around 2766 seconds was due to a cloud passing overhead.

Page 2



Graph 2: Temperature and Time for Entire Run

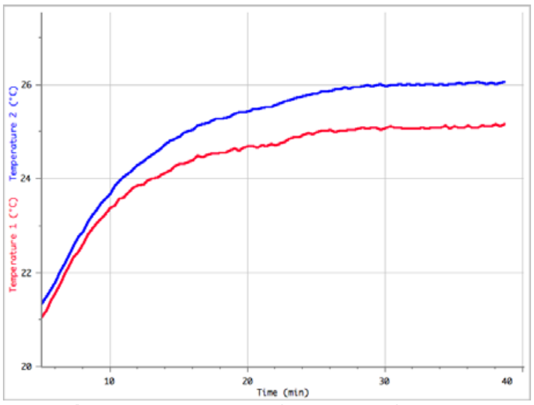


Graph 3: Equilibrium Temperatures

In Graph 3, the upper line is soda with CO_2 and the lower black line is flat soda. Soon after 3210 seconds both bottles reached equilibrium temperatures. The typical temperature difference was around 0.5 C. The slight fluctuation may be due to thermal noise, and does not really make any difference in my results.

This experiment was performed several times and the temperature variation ranged from 1 C° to about 0.5° difference. Graph 4 is an example where the difference was the greatest, one-degree, and it was performed on a different day.

The upper line in Graph 4 is soda with CO_2 and the lower red line is flat soda. In this run, I did not start recording temperature until the bottles were filled and set out in the sun. It was difficult to read the computer screen in bright sunlight, but the resulting graph clearly shows that the atmosphere in the $C0_2$ bottle retained more heat compared to the flat soda atmosphere.



Graph 4: Temperature and Time, Another Data Set

In **conclusion,** both bottles mimic the greenhouse effect, and the CO_2 bottle mimics the enhanced greenhouse effect—hence global warming—with a higher temperature, even at equilibrium, than the non-CO_2 bottle. My purpose was qualitative only. My physical model was for pedagogical uses, to illustrate with a hands-on approach the greenhouse effect. It was not to model planetary atmosphere but to provide a simple hand-on demonstration of the effects of greenhouse gases. The differences between the demonstration and planetary atmosphere are very complex but not relevant to this inquiry; errors and uncertainties need not be measured. However, sources of error include a slight agitating when filling the bottle to the flat bottle of soda that causes some CO_2; not starting at the same temperature (which is difficult); placing the two bottle in identical sunlight locations, with no shadow of one on the other; having the bottle on the same surface; and there may be some gas escaping from the lid and temperature probe connection. Further extensions might include recording data for several days (including the night time); building a larger container and adding living plants.

Although the hands-on model conclusions are exciting, a mathematical model is needed. We now turn to a mathematical model of the greenhouse effect.

A MATHEMATICAL MODEL OF THE GREENHOUSE EFFECT

Next I produced a Microsoft[5] *Excel* spreadsheet program using the physics equations of solar radiation and the relevant greenhouse effect equations. The **first spreadsheet investigation** (spreadsheet A) starts with the earth's temperature at 0°C and accepts the standard values of solar radiation, emissivity and albedo. I then progress in one-year steps to determine how long it takes for the earth's temperature to reach equilibrium.

The **second spreadsheet investigation** (spreadsheet B) continues this exploration by selecting a range of different starting temperature from very cold to very hot and then determines the time it takes to reach equilibrium.

The **third spreadsheet investigation** (spreadsheet C) varies of value of the earth's emissivity and then determines the time to reach equilibrium.

The **fourth spreadsheet investigation** (spreadsheet D) increases the $C0_2$ content and hence demonstrates the growing nature of the enhanced greenhouse effect and thus also demonstrates global warming. Unfortunately, the technical details here proved beyond the scope of my inquiry, and my results were unrealistic.

Technical Terms and Equations

Instead of footnoting each equation or numerical value I simply mention here that I used Wikipedia as a source of constant values and my IB physics textbook for the relevant equations.[6]

The equations involving **temperature** use the absolute or Kelvin scale but I graph the results using the Celsius scale. The conversion is straightforward.

$$T_{\text{Kelvin}} = T_{\text{Celsius}} + 273.15$$

Stefan-Boltzmann Constant (sigma, σ), is a constant of proportionality relating the total energy radiated per unit surface area of a black body in unit time; the S-F law states a proportionality to the fourth power of the thermodynamic temperature. The constant is:

$$\sigma = 5.67 \times 10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4}$$

The **solar constant** (K_{solar}) for the earth is the solar power (electromagnetic radiation) per unit area. It has an approximate but accepted value.

$$K_{\text{solar}} = 1367\ \text{W m}^{-2}$$

However, the **total power received by the earth** is proportional to the cross sectional area $\pi R^2_{\text{earth radius}}$. On average this power is distributed over the surface of the earth that is $4\pi R^2_{\text{earth radius}}$. To get the **average power per square meter** we therefore need to divide the solar constant by 4. This is explained on the Wikipedia web site for the solar constant. Hence I do this in the relevant equations, such as:

$$\frac{K_{\text{solar}}}{4} = \frac{1367\ \text{W m}^{-2}}{4} = 341.75\ \text{W m}^{-2}$$

At the surface of the earth, the **albedo** (α) is the ratio between the incoming radiation intensity and the amount reflected expressed as a coefficient or percentage. The value varies with surface material but an overall average ($\bar{\alpha}$) for the earth is given.

$$\bar{\alpha} = 0.31$$

The absorbed solar radiation per square meter I_{in} is therefore: $I_{\text{in}} = \left(1-\bar{\alpha}\right)\frac{K_{\text{solar}}}{4}$

$$I_{\text{in}} = \left(1-\bar{\alpha}\right)\frac{K_{\text{solar}}}{4} = \left(1-0.31\right)\frac{1367}{4} = 235.8075\ \text{W m}^{-2}$$

The **emissivity** (ε) of a material is the relative ability of its surface to emit energy by radiation. It is the ratio of energy radiated by a given material to the energy radiated by an ideal or black body at the same temperature. A true black body would have $\varepsilon = 1$ but for all other real bodies $\varepsilon < 1$. The emitted radiation of the earth thus depends on the average or emissivity, and can be taken as: $\bar{\varepsilon} = 0.612$

The **Stefan-Boltzmann Law** for a given surface area of one square meter can thus express the emitted radiation I_{out} of the earth on average. Temperature T is on the Kelvin scale.

$$I_{\text{out}} = \sigma \bar{\varepsilon} T^4$$

For the earth at a temperature of 0°C this gives an value of:

$$I_{\text{out (T = 0°C)}} = \left(5.67 \times 10^{-8}\right)\left(0.612\right)\left(273.15\right)^4 = 193.1698\ \text{W m}^{-2}$$

When the radiation intensity coming into the earth just equals the intensity going out, we have a state of **equilibrium.**

$$I_{\text{in}} = I_{\text{out}}$$

For the earth this equilibrium turns out to be about 14°C. We can solve for this as follows.

$$\left(1-\bar{\alpha}\right)\frac{K_{\text{solar}}}{4} = \sigma\bar{\varepsilon}T^4 \quad \rightarrow \quad T = \sqrt[4]{\frac{\left(1-\bar{\alpha}\right)K_{\text{solar}}}{4\sigma\bar{\varepsilon}}} = \sqrt[4]{\frac{\left(1-0.31\right)\left(1367\right)}{4\left(5.67\times10^{-8}\right)\left(0.612\right)}}$$

$$T = 287.1149\ \text{K} = 287.1149\ \text{K} - 273.15\text{°C} = 13.9649\text{°C} \approx 14.0\text{°C}$$

Page 5

When the input and output radiation are not in equilibrium then it is a **disturbed state**, and here the **net radiation absorbed** (I_{net}) (also in units of $\mathrm{W\,m^{-2}}$) is simply the difference between the incoming and outgoing energy intensities:

$$I_{net} = I_{in} - I_{out}$$

An example of the net radiation absorbed at time zero to one year is:

$$I_{net\ (T\,=\,0°C)} = 235.8075 - 193.1698 = 42.6377\ \mathrm{W\,m^{-2}}$$

The general equation for **surface heat capacity** is $Q = C_s A \Delta T$

The energy is *Q* and the surface area is *A* and the change in temperature is ΔT on the Kelvin scale, where *C* is the specific heat capacity. Here, however, we will understand **surface heat capacity per unit area**, hence the equation for the earth's surface heat capacity will be:

$$Q = C_{surface\ earth} \Delta T$$

Surface heat capacity of the earth is the heat required to raise the temperature of a unit area of a surface by one Kelvin, in units of watts years per square meter per Kelvin.

The **average global heat capacity** *C* has been estimated in terms of power for a year (recall that energy = power x time) for a unit area and a unit of temperature.

$$C_{earth\ surface\ one\ year} = 16.9\ \mathrm{W\,yr\,m^{-2}\,K^{-1}}$$

I now write an expression for the change in temperature per unit area over one year for the earth's surface as follows:

$$Q_{one\ year\ per\ unit\ area} = C_{earth\ surface} \Delta T$$

For the first year starting at *T* = 0°C we find the temperature change as follows:

$$\Delta T = \frac{Q_{one\ year\ per\ unit\ area}}{C_{surface}} = \frac{I_{net}\, t_{year}}{C_{surface}} = \frac{42.6377\ \mathrm{W\,m^{-2}} \times \text{one year}}{16.9\ \mathrm{W\,yr\,m^{-2}\,K^{-1}}} = 2.522940828\ \text{K change}$$

In the first year, with starting temperature 0 °C, the change in temperature would be about 2.5 K, so the new temperature would be 2.5 °C.

Excel Equations

Here are the equations I used in the spreadsheet.

Starting Temperature of 0°C on Kelvin scale in cell **D2: 273.15**

The year progression is generated by cell **C3: =C2+1**

$I_{(out)}$ in cell **E2: =(0.0000000567)*(0.612)*(D2+273.15)^4**

I_{out} at starting temperature of 0°C is:

$$I_{out} = \sigma \bar{\varepsilon} T^4 = \left(5.67 \times 10^{-8}\right)\left(0.612\right)\left(273.15\right)^4 = 193.1698 \text{ W m}^{-2}$$

$I_{(net)}$ in cell **F2: =(235.8075)-E2**

I_{net} at starting temperatures of 0°C is:

$$I_{net} = I_{in} - I_{out} = 235.8075 - 193.1698 = 42.6377 \text{ W m}^{-2}$$

The change in **temperature *T*** at year intervals is calculated as **D3: =D2+(F2/16.9)**

At the end of the first interval this is equal to:

$$\Delta T = 0°\text{C} + \left(\frac{I_{net}}{C}\right) = 0 + \frac{42.6377}{16.9} = 2.5229408 \text{ C}°$$

$$T = T_{start} + \Delta T = 0 + 2.5229408 \text{ C}° \approx 2.52 \text{ °C}$$

Spreadsheet A—Time to Reach Equilibrium

The following is the textbook model of the earth. The solar constant is 1367 W m^{-2} and the albedo ratio is 0.31 with an emissivity of 0.613. The earth's surface heat capacity is taken as 16.9 W yr m^{-2} K^{-1}.

The data runs from a starting temperature of 0°C for 60 years, more than enough time to find the equilibrium temperature. Here (on the right) is a sample of my data.

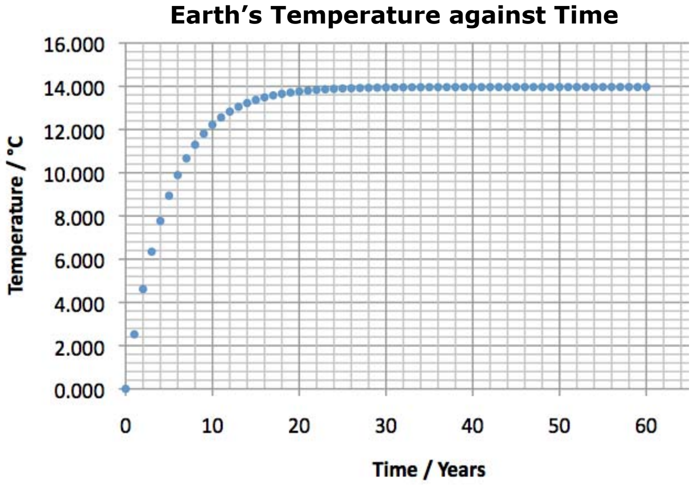
Physics Climate Model

Solar Constant	Year	Temperature	$I_{(out)}$	$I_{(net)}$
1367	0	0.000	1.9317E+02	4.2638E+01
	1	2.5229E+00	2.0041E+02	3.5401E+01
Albedo	2	4.6177E+00	2.0657E+02	2.9240E+01
0.31	3	6.3479E+00	2.1176E+02	2.4045E+01
	4	7.7707E+00	2.1611E+02	1.9700E+01
Stef-Boltz Constant	5	8.9364E+00	2.1972E+02	1.6091E+01
5.67E-08	6	9.8885E+00	2.2270E+02	1.3109E+01
	7	1.0664E+01	2.2515E+02	1.0658E+01
Emissivity	8	1.1295E+01	2.2716E+02	8.6501E+00
0.612	9	1.1807E+01	2.2880E+02	7.0107E+00
	10	1.2222E+01	2.3013E+02	5.6755E+00
Heat Capacity	11	1.2557E+01	2.3122E+02	4.5903E+00
16.9	12	1.2829E+01	2.3210E+02	3.7098E+00
	13	1.3048E+01	2.3281E+02	2.9963E+00
Absorbed $I_{(in)}$	14	1.3226E+01	2.3339E+02	2.4189E+00
235.8075	15	1.3369E+01	2.3386E+02	1.9520E+00
	16	1.3484E+01	2.3423E+02	1.5746E+00
Start Temp = 0°C	17	1.3578E+01	2.3454E+02	1.2699E+00
	18	1.3653E+01	2.3478E+02	1.0240E+00
	19	1.3713E+01	2.3498E+02	8.2551E-01
	20	1.3762E+01	2.3514E+02	6.6542E-01
	21	1.3802E+01	2.3527E+02	5.3631E-01
	22	1.3833E+01	2.3538E+02	4.3222E-01
T = 14.0°C for $I_{in}=I_{out}$	23	1.3859E+01	2.3546E+02	3.4831E-01
	24	1.3879E+01	2.3553E+02	2.8067E-01
$T_K = T_C + 273.15$	25	1.3896E+01	2.3558E+02	2.2615E-01
	26	1.3909E+01	2.3563E+02	1.8222E-01
	27	[illegible]	[illegible]	[illegible]

And on for 60 years.

	Year	Temperature	$I_{(out)}$	$I_{(net)}$
	53	1.3965E+01	2.3581E+02	5.3240E-04
	54	1.3965E+01	2.3581E+02	4.2890E-04
	55	1.3965E+01	2.3581E+02	3.4553E-04
	56	1.3965E+01	2.3581E+02	2.7836E-04
	57	1.3965E+01	2.3581E+02	2.2425E-04
	58	1.3965E+01	2.3581E+02	1.8066E-04
	59	1.3965E+01	2.3581E+02	1.4554E-04
	60	1.3965E+01	2.3581E+02	1.1725E-04

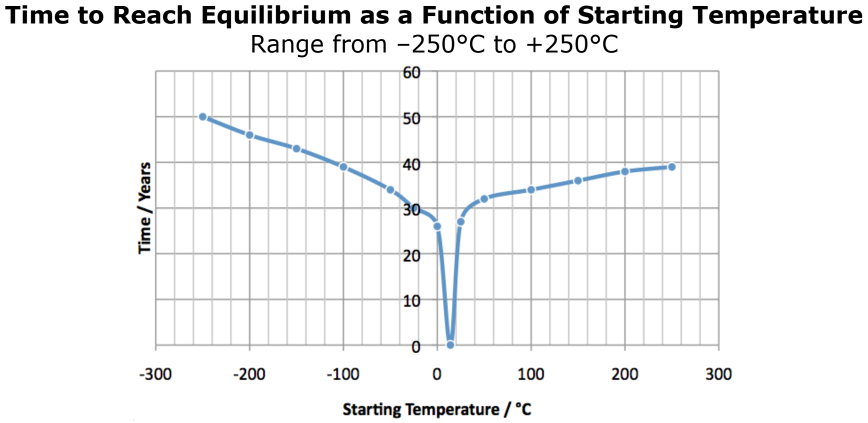
Here is the graph of the earth's temperature as a function of time based on the above model.



The time to reach equilibrium is approximately 25 years. When looking at three significant figures, the temperature of 13.9°C is reached after 26 year. Looking at the equilibrium temperature to three decimal places, however, it takes 43 years to reach 13.964°C.

Spreadsheet B—Different Starting Temperatures and Equilibrium Time

In the next investigation I varied the initial temperature from −250°C to +250°C and then I determined the relationship between the starting temperature of the earth and the number of years it took to reach equilibrium. The results are interesting.

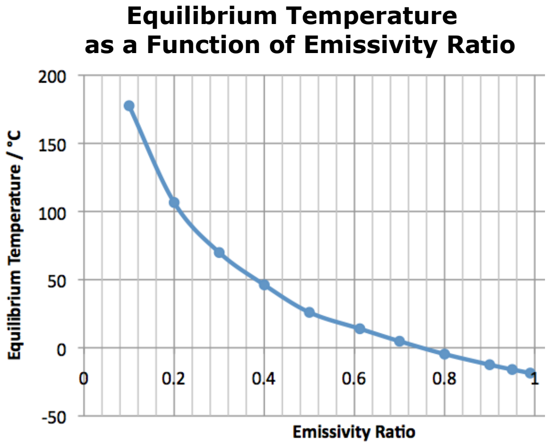


This graph indicates the time in years to reach equilibrium with different starting temperatures. Given the initial parameters for the earth, the natural equilibrium is around 14°C so a starting temperature at 14°C would require no time to reach equilibrium. The actual equilibrium temperature has been calculated to be 13.965 °C. It is interesting to note that the curve is not symmetrical on either side of the equilibrium position. In one case the earth is warming up and the other it is cooling down.

Page 8

Spreadsheet C—Different Emissivity Values and Equilibrium Temperature

In the next investigation I varied the emissivity ratio from low to high and then I revealed a relationship between emissivity and equilibrium temperature.



In the above graph, the emissivity ratio ranges from 0.10 to 0.99 revealing an equilibrium range from 177°C to −18.6°C. This is what you would expect: a decreasing equilibrium temperature as more of the intensity of incident radiation is reflected outwards.

Spreadsheet D—Adding CO_2 to the Model

In my last investigation I made the model more realistic. My other mathematical models were simplified, of course, when compared to the real world. First, they are one-dimensional whereas the earth's atmosphere is three-dimensional; second, they took steps on one year intervals whereas in the real world the process is continuous; and third, most importantly, my model assumed greenhouse gases were constant, which they are not.

A more realistic model would add a factor for the increasing $C0_2$ and other greenhouse gases as a function of time. I should add a factor to account for the every increasing rate of CO_2. The net result would be a higher equilibrium temperature, and a more dynamic process. We are told in a Wikipedia article that if the CO_2 level were to double then the temperature would increase by +3K.

There is a mathematical factor called "radiative forcing" that increases the rate of the greenhouse gases. The equation is only an approximation to the first order, but it would be interesting to use this in my model. The equation is:

$$\Delta T_{\text{kelvin}} = \lambda \Delta F = \lambda\left(5.35 \ln \frac{C}{C_0}\right)$$

where $\lambda = 0.8$ (a proportionality constant for the earth) and C is the CO_2 concentration and C_0 is the concentration reference. See the Carbon Dioxide Information Analysis Center

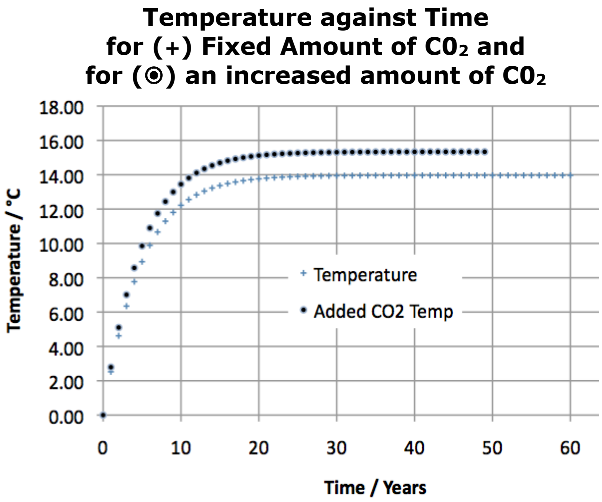
Page 9

(CDIAC) online at http://cdiac.ornl.gov/pns/current_ghg.html and see the Wikipedia article on Radiative Forcing.

Here is one example of a calculation. Over 250 years ago, in 1750, the CO_2 content was about 280 parts per million, and today it is 388.5 ppm. The ratio or the increase is thus: 1.3875, and so the temperature change over this period is about 1.4 K.

$$\Delta T = \lambda \Delta F = (0.8)\left[5.35\ln\left(\frac{C}{C_0}\right)\right] = (0.8)\left[5.35\ln\left(\frac{388.5}{280}\right)\right] = 1.4017\,\mathrm{K}$$

Here is my graph of temperature against time with enhanced greenhouse gas.



My original model had a fixed value for CO_2, and had an equilibrium temperature of 13.96°C. See the cross (+) data points on the graph. They represent my original data. When I added a factor that increased CO_2 each year, the equilibrium temperature was higher, this time at 15.34°C. See the black circle data points on the graph. This does not compare to the accepted value of temperature increase, which over the past 100 year was +0.8 °C. My model had an increase of 1.38 K, way too much as it represents a CO_2 increase of about 50%. My model needs serious work. However, this last part of my mathematical model is left for future studies.

Footnotes

(1) Various textbooks and web sites were consulted for the general information in this study. The same or similar details can be found from many different sources. Here are the main sources of information I consulted for this study.

"Elementary Climate Physics" by F. W. Taylor (Oxford University Press, 2006), Chapters 1 and 7.

"Physics for the IB Diploma" by K.A. Tsokos (5th edition, Cambridge University Press), Topic 7.

"Physics for use with the IB Diploma Programme" by G. Kerr and P. Ruth, (3rd edition, IBID Press), Chapter 8.

http://www.esrl.noaa.gov/gmd/outreach/lesson_plans/Modeling%20the%20Greenhouse%20Effect.pdf

http://passporttoknowledge.com/scic/greenhouseeffect/educators/greenhouseeffect.pdf http://www.espere.net/Unitedkingdom/water/uk_watexpgreenhouse.htm

http://www.wested.org//earthsystems/energy/greenhouse.html

(2) The best Internet simulation for the greenhouse effect can be found at the University of Colorado at Boulder web site for Physics Education Technology, PhET. **http://phet.colorado.edu/**

Other simulations include: http://earthguide.ucsd.edu/earthguide/diagrams/greenhouse/ and http://epa.gov/climatechange/kids/global_warming_version2.html

(3) My initial idea came from a slightly different experiment but one using a similar technique and this can be found at http://www.ucar.edu/learn/1_3_2_12t.htm. Further research revealed two excellent sources of technical help and ideas: "Greenhouse Effect Study Apparatus," *American Journal of Physics*, Volume 41, #442, March 1973, and "A Simple Experiment to Demonstrate the Effects of Greenhouse Gases" by C. F. Keating in *The Physics Teacher* Volume 45, September 2007, pages 376 to 378.

(4) Vernier hardware and software information can be found at http://www.vernier.com/. Note that I used the Surface Temperature Sensor and not the Temperature Probe because the surface sensor is better suited for low-density measurements, like air.

(5) http://office.microsoft.com/en-us/excel/

(6) For numerical values with as many as possible decimal places, I used *Wikipedia* as a source (http://en.wikipedia.org/) and my textbook "Physics for the IB Diploma" by K.A. Tsokos (Cambridge Press) for the relevant equations.